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Kantowski–Sachs universe

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Abstract
We consider the brane Kantowski–Sachs universe when bulk space is five-dimensional anti-de Sitter space. The corresponding cosmological equations with perfect fluid are written. For several specific choices of relation between energy and pressure, the behaviour of scale factors at early time is found. In particular for $\gamma = 3/2$, Kantowski–Sachs brane cosmology is modified to become the isotropic one, while for $\gamma = 1$ it remains the anisotropic cosmology in the process of evolution.

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1. Introduction

Since the work by Randall–Sundrum [1], it was realized that brane cosmology is quite similar to the standard four-dimensional cosmology at late times. However, it may significantly differ from the standard cosmology at early times. Nevertheless, the brane gravity shows Newtonian behaviour despite the fact that braneworld is five-dimensional.

In standard as well as in brane cosmology it is quite possible that the early universe could be the anisotropic one. During the evolution, the anisotropy stage should quickly be changed to the isotropic stage due to classical or quantum matter effects, or due to the modification of the gravity theory or by some other phenomena. In the present paper we discuss the brane Kantowski–Sachs (KS) cosmology and compare it with the standard Kantowski–Sachs cosmology in Einstein gravity for some specific matter choice. It is shown that the brane KS cosmology may quickly become isotropic for some choices of matter.

2. Standard versus brane Kantowski–Sachs cosmology

2.1.

We start from the five-dimensional braneworld which is defined by the condition $Y(X^I) = 0$, where $I = 0, 1, 2, 3, 4$ are five-dimensional coordinates. The starting action in the five-dimensional space is [2, 3]:

$$S = \int \mathrm{d}^5 X \sqrt{-g_5} \left(\frac{1}{2k_5^2} R_5 - \Lambda_5 \right) + \int_{Y=0} \mathrm{d}^4 x \sqrt{-g} \left(\frac{1}{k_5^2} K^{\pm} - \lambda + L^m \right), \tag{1}$$

with $k_5^2 = 8\pi G_5$ being the five-dimensional gravitational coupling constant and x^μ, ($\mu = 0, 1, 2, 3$) the induced four-dimensional brane coordinates. R_5 is the 5D intrinsic curvature in the bulk and $K^{\pm}$ is the intrinsic curvature on either side of the brane.

The 5D Einstein equation has the form

$$^{(5)}G_{IJ} = k_5^2 \, ^{(5)}T_{IJ}, \qquad ^{(5)}T_{IJ} = -\Lambda_5 {}^{(5)}g_{IJ} + \delta(Y)\left[-\lambda g_{IJ} + T^m_{IJ}\right].$$

Assuming a metric of the form $\mathrm{d}s^2 = (n_I n_J + g_{IJ})\,\mathrm{d}x^I\,\mathrm{d}x^J$, with $n_I\,\mathrm{d}x^I = \mathrm{d}\xi$ the unit normal to the $\xi =$ const hypersurfaces and g_{IJ} the induced metric on $\xi =$ const hypersurfaces, the effective four-dimensional gravitational equations on the brane take the form [2–4]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^4 S_{\mu\nu} - E_{\mu\nu}, \tag{2}$$

where

$$S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T^\alpha_\mu T_{\alpha\nu} + \frac{1}{24} g_{\mu\nu} (3T^{\alpha\beta} T_{\alpha\beta} - T^2),$$

and $\Lambda = k_5^2(\Lambda_5 + k_5^2\lambda^2/6), k_4^2 = k_5^2\lambda^2/6$ and $E_{IJ} = C_{IAJB}n^A n^B$. C_{IAJB} is the five-dimensional Weyl tensor in the bulk and λ is the vacuum energy on the brane. $T_{\mu\nu}$ is the matter energy–momentum tensor on the brane with components $T^0_0 = -\rho, T^1_1 = T^2_2 = T^3_3 = p$ and $T = T^\mu_\mu$ is the trace of the energy–momentum tensor. One chooses $p = (\gamma - 1)\rho$, hence $1 \leqslant \gamma \leqslant 2$.

From equation (2) it follows that there exist several cases:

(1) conventional Einstein theory (CET) which is 4D and

(2) brane cosmology (BC) which includes 5D effects.

These cases originate from different relations between CET and braneworld scenario which can lead to two types of corrections: (a) the matter fields contribute local 'quadratic' energy–momentum correction via the tensor $S_{\mu\nu}$ and (b) the 'nonlocal' effects via bulk Weyl tensor.

We will consider the brane metric in the Kantowski–Sachs form [5, 6]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_1(t)^2 dr^2 + a_2(t)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2).$$

The following variables are convenient to introduce:

$$V = a_1 a_2^2, \tag{3}$$

$$H_i = \frac{\dot{a}_i}{a_i}, \qquad H = \frac{1}{3}(H_1 + 2H_2) = \frac{\dot{V}}{3V}. \tag{4}$$

2.2.

In this case the Einstein equation is

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu}, \qquad \nabla_\mu T^{\mu\nu} = 0, \tag{5}$$

with $G_{\mu\nu}$ being the Einstein tensor (4D), Λ the cosmological constant, k_4 the gravitational coupling, $k_4^2 = 8\pi G$. These equations for Kantowski–Sachs universe become

$$\begin{aligned}
&\frac{\dot{a}_2^2}{a_2^2} + 2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} = \Lambda + k_4^2\rho,\\
&2\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda + k_4^2\rho(1-\gamma),\\
&\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} = \Lambda + k_4^2\rho(1-\gamma),\\
&\dot{\rho} + \gamma\rho\left(\frac{\dot{a}_1}{a_1} + 2\frac{\dot{a}_2}{a_2}\right) = 0.
\end{aligned} \tag{6}$$

Equation (6) can be easily solved to describe the time evolution law of the energy density of the fluid:

$$\rho = \rho_0 V^{-\gamma}, \qquad \rho_0 = \text{constant} > 0.$$

One can rewrite this equation in another form:

$$\frac{\mathrm{d}}{\mathrm{d}t}(VH_1) = \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma), \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(VH_2) = \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) - a_1, \tag{8}$$

$$3\dot{H} + H_1^2 + 2H_2^2 = \Lambda + \frac{1}{2}k_4^2\rho_0 V^{-\gamma}(2-3\gamma). \tag{9}$$

From the first two equations, the equation for V may be written as

$$\ddot{V} = 3\Lambda V + \tfrac{3}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) - 2a_1. \tag{10}$$

The equation can be partly integrated:

$$\dot{V} = \sqrt{3\Lambda V^2 + 3k_4^2\rho_0 V^{2-\gamma} - 2\int a_1\,\mathrm{d}V}. \tag{11}$$

From (7) and (8) it follows that

$$H_1 = H + \frac{2}{3V}K, \qquad H_2 = H - \frac{1}{3V}K, \qquad K = \int a_1\,\mathrm{d}t.$$

If we substitute these equations into (9) then

$$-2a_1V + \frac{4}{3}\int a_1\,\mathrm{d}V + \frac{2}{3}\left(\int a_1\,\mathrm{d}t\right)^2 = 0. \tag{12}$$

Let us consider the asymptotic behaviour. If V is large then solution has the simple form

$$V = a\,\mathrm{e}^{\sqrt{3\Lambda}t}, \qquad H_1 = H_2.$$

Using definitions (3) and (4), we obtain

$$a_1 = \frac{a}{b^2}\,\mathrm{e}^{\sqrt{\Lambda/3}t}, \qquad a_2 = b\,\mathrm{e}^{\sqrt{\Lambda/3}t}, \tag{13}$$

where a and b are constants, Λ is not zero. The behaviour of the system does not depend on γ and the anisotropic CET universe becomes isotropic one for large V due to classical matter effects.

The situation for extremely small V is different. The properties of the CET universe will depend on the value of γ. From equation (11)

$$\dot{V} = \sqrt{3k_4^2\rho_0 V^{2-\gamma} + b}. \tag{14}$$

For $\gamma = 3/2$, the solution takes the form

$$a_1 = \frac{9a^{2/3}(t-t_0)^{4/9}}{4c^2}, \qquad a_2 = c(t-t_0)^{4/9}, \qquad V = \frac{9}{4}a^{2/3}(t-t_0)^{4/3}. \tag{15}$$

Here, c is the constant of integration and $a = \frac{1}{2}k_4^2\rho_0$ also is supposed, that $V(t_0) = 0$. For $\gamma = 1$ the solution cannot be found in analytical form; however we can find the asymptotic solution at small values of V:

$$V \sim (t-t_0)^2, \qquad a_1 \sim (t-t_0)^{2/3}, \qquad a_2 \sim (t-t_0)^{2/3}. \tag{16}$$

It is visible that in this case a solution is also isotropic. However, it is necessary to note that at increased t this isotropy will be broken. But at sufficiently large values of V the solution becomes isotropic again.

Hence, we finished the review of the anisotropic CET universe. It is found that even for small cosmological time the process of isotropization starts quickly.

2.3.

In this subsection, we consider the brane cosmology with zero Weyl tensor which is natural for 5D AdS bulk. The Einstein equations and evolution law of the energy density take the form

$$\begin{aligned}
&\frac{\dot{a}_2{}^2}{a_2^2} + 2\frac{\dot{a}_1\dot{a}_2}{a_1a_2} + \frac{1}{a_2^2} = \Lambda + k_4^2\rho + \frac{1}{12}k_5^4\rho^2, \\
&2\frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_2{}^2}{a_2^2} + \frac{1}{a_2^2} = \Lambda + k_4^2\rho(1-\gamma) + \frac{1}{12}k_5^4\rho^2(1-2\gamma), \\
&\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1\dot{a}_2}{a_1a_2} = \Lambda + k_4^2\rho(1-\gamma) + \frac{1}{12}k_5^4\rho^2(1-2\gamma), \\
&\dot{\rho} + \gamma\rho\left(\frac{\dot{a}_1}{a_1} + 2\frac{\dot{a}_2}{a_2}\right) = 0.
\end{aligned} \tag{17}$$

Equation (17) can be easily solved:

$$\rho = \rho_0 V^{-\gamma}, \qquad \rho_0 = \text{constant} > 0.$$

We can rewrite the above equations in another form:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(VH_1) = \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) + \frac{1}{12}k_5^4\rho_0^2 V^{1-2\gamma}(1-\gamma), \\
&\frac{\mathrm{d}}{\mathrm{d}t}(VH_2) = \Lambda V + \frac{1}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) + \frac{1}{12}k_5^4\rho_0^2 V^{1-2\gamma}(1-\gamma) - a_1, \\
&3\dot{H} + H_1^2 + 2H_2^2 = \Lambda + \frac{1}{2}k_4^2\rho_0 V^{-\gamma}(2-3\gamma) + \frac{1}{12}k_5^4\rho_0^2 V^{-2\gamma}(1-3\gamma).
\end{aligned}$$

From the first two equations the equation for V can be obtained:

$$\ddot{V} = 3\Lambda V + \frac{3}{2}k_4^2\rho_0 V^{1-\gamma}(2-\gamma) + \frac{1}{4}k_5^4\rho_0^2 V^{-2\gamma}(1-\gamma) - 2a_1. \tag{18}$$

It is easy to show that from these equations we can formally get the same equations as in conventional Einstein theory:

$$H_1 = H + \frac{2}{3V}K, \qquad H_2 = H - \frac{1}{3V}K,$$

$$K = \int a_1\,\mathrm{d}t, \qquad -2a_1V + \frac{4}{3}\int a_1\,\mathrm{d}V + \frac{2}{3}\left(\int a_1\,\mathrm{d}t\right)^2 = 0.$$

The asymptotic behaviour for large V has the same form as in CET. However, for extremely small V one obtains the difference in comparison with CET.

For $\gamma = 1$

$$a_1 = \frac{1}{c_2^2}\left(c_1 + \sqrt{a^2+bt}\right)^{1/3-\frac{2\sqrt{b}}{3\sqrt{a^2+b}}},$$

$$a_2 = c_2\left(c_1 + \sqrt{a^2+bt}\right)^{1/3+\frac{\sqrt{b}}{3\sqrt{a^2+b}}},$$

$$V = c_1 + \sqrt{a^2+bt}.$$

Thus, unlike the situation with CET, the analytical solution appears. Nevertheless, the solution remains anisotropic for KS cosmology.

For $\gamma = 3/2$

$$a_1 = \frac{1}{c_2^2}\left(\frac{3a}{2}\right)^{2/3}(t-t_0)^{2/9},$$

$$a_2 = c_2(t-t_0)^{2/9},$$

$$V = \left(\frac{3a}{2}\right)^{2/3}(t-t_0)^{2/3}.$$

Here b, t_0, c_2 are constants, $a^2 = \frac{1}{4}k_5^4\rho_0^2$.

3. Discussion

In summary, we compared the KS brane cosmology with the KS cosmology for Einstein theory for several classical matter choices.

We demonstrated that in contrast to CET with $\gamma = 1$, brane KS universe remains anisotropic. For $\gamma = 3/2$, brane KS universe becomes isotropic at early times analogous to the CET case. However, the details of isotropization (scale factors) are slightly different, which shows the role of five-dimensional bulk.

It is possible to show that the asymptotical behaviour of a solution at $\gamma = 2$ will be similar to the case $\gamma = 3/2$—it will be isotropic in both models (CET and brane cosmology).

One can also consider the simple case, when $a_1 = \text{const}$. For CET we get the exact solution

$$a_2 = \pm\sqrt{\frac{k_4^2\rho_0}{a_1^2} - t^2 + 2tc - c^2}.$$

Here c is an integration constant, $\Lambda = 0, \gamma = 2$. However, for brane cosmology there is no exact solution. This shows the qualitative difference between CET and brane cosmology.

It would be really interesting to understand the role of quantum effects like in brane new world scenario [7] in the above KS brane cosmology. In particular, it is expected that quantum effects may lead to isotropic cosmology even for an initially anisotropic brane universe as happens for standard Einstein theory [8]. Another interesting topic could be the generalization of the discussion for brane wormholes (see [9] for a recent discussion).

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